Exercise 9.4

\[ f(x) = \theta x^{\theta-1} \quad 0 < x < 1 \]

A sample of size 3 is collected \( X_1 = 0.4 \), \( X_2 = 0.7 \), \( X_3 = 0.9 \). Estimate \( \theta \).

One unknown, with method of moments, we may use \( M_1 \):

\[ M_1 = E(x) = \int_0^1 xf(x) dx = \int_0^1 x \theta x^{\theta-1} dx = \int_0^1 \theta x^\theta dx \]

\[ = \left. \theta x^{\theta+1} \right|_0^1 \bigg|_{x=0}^{x=1} = \frac{\theta}{\theta+1} \]

\[ M_1 = \bar{X} = \frac{0.4 + 0.7 + 0.9}{3} = \frac{2}{3} \]

\[ m_1 = m_2 \Rightarrow \frac{2}{3} = \frac{\theta}{\theta+1} \Rightarrow \hat{\theta} = 2 \quad \text{(method of moments)} \]

Let's now try maximum likelihood.

Probability of observing the sampling units

\[ \prod_{i=1}^{3} \theta x_i \]

Taking the logarithm of that probability, we get:

\[ \frac{3}{\theta} \ln \theta + (\theta - 1) \sum_{i=1}^{3} \ln x_i = 3 \ln \theta + (\theta - 1) \sum_{i=1}^{3} \ln x_i \]

Take derivative with respect to \( \theta \) and equate to 0:

\[ \frac{3}{\theta} + \sum_{i=1}^{3} \frac{1}{x_i} = 0 \Rightarrow \hat{\theta} = -\frac{3}{\sum_{i=1}^{3} \ln x_i} \approx 2.18 \]
Exercise 9.5

The given distribution has one unknown parameter: \( \Theta \)

(a) We may use the first moment \( M_1 \) to estimate it

\[
M_1 = E(x) = \int_0^\infty x f(x) \, dx = \int_0^\infty x \left( \frac{1}{\Theta} e^{-x/\Theta} + \frac{1}{10} e^{-x/10} \right) \, dx
\]

We need to use the mathematical knowledge that

\[
\int_0^\infty x e^{-x/a} \, dx = -a e^{-x/a}
\]

By using this, we can write \( M_1 \) as

\[
M_1 = \left. \frac{1}{2\Theta} e^{-x/\Theta} (x+\Theta) + \frac{1}{20} e^{-x/10} (x+10) \right|_{x=0}^{x=\infty}
\]

\[
= \left. \frac{-(x+\Theta)}{2} e^{-x/\Theta} + \frac{-2}{2} e^{-x/10} \right|_{x=0}^{x=\infty}
\]

\[
= 0 - \left( -\frac{\Theta}{2} - \frac{10}{2} \right) = \frac{\Theta}{2} + 5
\]

\[
\Rightarrow M_1 = \frac{\Theta}{2} + 5
\]

\[
M_1 = \bar{X} = 150/10 = 15
\]

\[
\Rightarrow \frac{\Theta}{2} + 5 = 15 \Rightarrow \frac{\Theta}{2} = 10 \Rightarrow \Theta = 20
\]

(b) \( \hat{\Theta} + 5 = \bar{X} \Rightarrow \hat{\Theta} = 2\bar{X} - 10 \)

\[
\sigma(\hat{\Theta}) = 2\sigma(\bar{X}) = \frac{2\sigma(x)}{\sqrt{n}}
\]
(b) To find $\sigma(x)$ we need to also find $E(x^2) = \int_0^\infty x^2 f(x) \, dx$.

We need to use the mathematical knowledge that

$$\int x^2 e^{-x/a} \, dx = -a e^{-x/a} \left( x^2 + 2ax + 2a^2 \right) \uparrow\text{again 0 when } x=\infty \uparrow 1 \text{ when } x=0$$

$$E(x^2) = \frac{1}{2} (2\Theta^2 + 2 \cdot 10^2) = \Theta^2 + 100$$

$$Var(x) = E(x^2) - E(x)^2 = \Theta^2 + 100 - \left( \frac{\Theta}{2} + 5 \right)^2$$

$$= 0.75\Theta^2 - 5\Theta + 75$$

$\hat{\Theta} = 20$ and $n=10$ in our example.

$\Rightarrow$ Standard error $= \sigma(\hat{\Theta}) = \frac{2 \sigma(x)}{\sqrt{n}} = 2 \sqrt{\frac{0.75(20)^2 - 5 \cdot 20 + 75}{n}}$

$= 5.24$
Exercise 9.7

In this question $\bar{x} = 37.7$ and $\sigma(\bar{x})$ (i.e. standard error) is already given as 9.2.

For part (a) all we have to do is look up $z_{\alpha/2}$

$z_{0.1} = z_{0.05} = 1.645$

(a) confidence interval

$\bar{x} \pm 1.645 \cdot 9.2 = [22.566, 52.834]$

** If the given standard deviation is the standard deviation of concurrent users at a given time, i.e., $\sigma(\bar{x})$,
then we need to build the confidence interval as

$\bar{x} \pm 1.645 \cdot \frac{9.2}{\sqrt{100}} = [36.1866, 39.2134]$

(b) $H_0: M_0 = 35$ $H_A: M > 35$ one-sided right tail test

$z_{0.01} = 2.326$

Test statistic

$z = \frac{\bar{x} - M_0}{\sigma(\bar{x})} = \frac{37.7 - 35}{9.2} = 0.2935$

$Z$ is not $> z_{0.01}$ so cannot reject $H_0$.

Not sufficient evidence.

** If $\sigma(\bar{x}) = 9.2$, then $\sigma(x) = \frac{9.2}{\sqrt{100}} \Rightarrow z = 2.9$ then we can reject $H_0$. 
Exercise 9.16

\[ \hat{p}_1 = \frac{10}{250} = 0.04 \quad \hat{p}_2 = \frac{18}{300} = 0.06 \]

\[ n_1 = 252 \quad n_2 = 300 \]

Estimated sample variance for the first sample

\[ \frac{\hat{p}_1 (1-\hat{p}_1)}{n_1} = \frac{0.04 \cdot 0.96}{250} = 0.0001536 \]

Estimated sample variance for the second sample

\[ \frac{\hat{p}_2 (1-\hat{p}_2)}{n_2} = \frac{0.06 \cdot 0.94}{300} = 0.000188 \]

\[ \text{Var}(\hat{p}_1 - \hat{p}_2) = \text{Var}(\hat{p}_1) + \text{Var}(\hat{p}_2) \]

\[ \Rightarrow \text{standard error of difference} = \sqrt{0.0001536 + 0.000188} \]

\[ = 0.01848 \]

(a) \( \alpha = 0.02 \quad Z_{\alpha/2} = 2.326 \)

confidence interval

\[ 0.04 - 0.06 \pm 2.326 \cdot 0.01848 \]

\[ = [-0.063, 0.023] \]

(b) Two-tail test \( H_0: p_1 = p_2 \quad H_A: p_1 \neq p_2 \) \( \alpha \) is same as (a)

Confidence interval contains 0, therefore cannot reject \( H_0 \).
Exercise 9.17

At 95% confidence \( \alpha = 0.05 \) \( n = 900 \)

\( z_{\alpha/2} = 1.96 \)

The margin of error for \( \hat{p}_1 = 0.45 \) at 95% confidence level is

\[
1.96 \sqrt{\frac{0.45 \cdot 0.55}{900}} = 0.0325
\]

In other words, 45% is

45 + 3.25% at 95% confidence level

The margin of error for \( \hat{p}_2 = 0.35 \) at 95% confidence level is

\[
1.96 \sqrt{\frac{0.35 \cdot 0.65}{900}} = 0.0312
\]

In other words, 35 is

35 + 3.12% at 95% confidence level

The margin of error for \( \hat{p}_1 - \hat{p}_2 = 0.10 \) at 95% confidence level

\[
1.96 \sqrt{\frac{0.45 \cdot 0.55}{900} + \frac{0.35 \cdot 0.65}{900}} = 0.045
\]

In other words, \( \%10 \) difference is

10 + 4.5% at 95% confidence level.