CENG 222
Statistical Methods for Computer Engineering

Spring 2020-2021

Section 1
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Online Materials will be available on ODTU-Class
Goals of the course

• Learn techniques and tools to be able to:
  – analyze and interpret large scale data,
  – apply probability theory and statistics to handle uncertainty,
  – infer facts and relationships from collected data, and
  – construct simulations by sampling from arbitrary distributions

• Acquire skills for the hot new CS field: “Data Science”
Course outline

- See the syllabus on ODTU-Class
Grading

• Midterm exam – 30%
• Final exam – 30%
• 4 Assignments – 32%
• 12 Quizzes (best 8 of 12) – 8 %
ODTU Class

- Syllabus
- Course textbook
- Lecture slides and reading materials
- Assignments
- Grades
COW

• A forum for course related discussions
Textbook

- Your main resource of study for this course
Probability

• Studies uncertainty
• A random experiment
  – An experiment/observation which does not have a certain outcome before it is conducted
  • Examples
    – Tossing a coin
    – Observing the life time of a light bulb
    – Number of games the Cavaliers will win this season
    – Others?
Sample space

• The set of all possible outcomes of a random experiment is called the sample space
  – Tossing a coin:
    • Sample space = \{H, T\}
  – Tossing two coins:
    • Sample space = \{HH, HT, TH, TT\}
  – Lifetime of a light bulb:
    • Sample space = [0, +\infty)
Event

• Any collection of possible outcomes of an experiment
  – Any subset of the sample space

• Examples:
  – Experiment: tossing two coins. Event: obtaining exactly one head. \(\{HT, TH\} \subset \{HH, HT, TH, TT\}\)
  – Experiment: lifetime of light bulb. Event: light bulb does not last more than a month. \([0, 1] \subset [0, +\infty)\)
Event

- A sample space of $N$ possible outcomes yields $2^N$ possible events
- Example: tossing a dice once
  - Sample space = \{1,2,3,4,5,6\}
  - Number of possible events = $2^6 = 64$
- Example events?
Notation used in the book

- $\Omega = $ sample space
- $\emptyset = $ empty event
- $P\{E\} = $ probability of event $E$
Event algebra

• Union of two events: same as set union
  – \( A \cup B = \{ x : x \in A \text{ or } x \in B \} \)

• Intersection of two events: same as set intersection
  – \( A \cap B = \{ x : x \in A \text{ and } x \in B \} \)

• Complementation: same as in sets
  – \( A^c \) or \( \bar{A} = \{ x : x \in \Omega \text{ and } x \notin A \} \)

• Difference: same as in sets
  – \( A \setminus B = \{ x : x \in A \text{ and } x \notin B \} \)
Disjoint and exhaustive events

- Disjoint events: If $A$ and $B$ have no outcomes in common, i.e., $A \cap B = \emptyset$
  - Also called mutually exclusive events
- If the union of a number of events equals the sample space, they are called exhaustive
  - $A \cup B \cup C = \Omega$
Complement, Union, Intersection

- \( A \cup B = \overline{A} \cap \overline{B} \)
- \( A \cap B = \overline{A} \cup \overline{B} \)
- \( E_1 \cup E_2 \cup E_3 \cup E_4 = \overline{E_1} \cap \overline{E_2} \cap \overline{E_3} \cap \overline{E_4} \)
- \( E_1 \cap E_2 \cap E_3 \cap E_4 = \overline{E_1} \cup \overline{E_2} \cup \overline{E_3} \cup \overline{E_4} \)
Probability

• Assignment of a real number to an event
  – The relative frequency of occurrence of an event in a large number of experiments

• $P(A)$

• Axioms of probability:
  – $P(A) \geq 0$
  – $P(\Omega) = 1$
  – If $A$ and $B$ are mutually exclusive events, then $P(A \cup B) = P(A) + P(B)$

• Any function that satisfies these axioms is called a probability function
Example

• Experiment:
  – Tossing two coins
  – $A = \{\text{obtaining exactly one head}\}$
  – $P(A) = ?$
Computing probabilities

• for non-“mutually exclusive” events:
  \[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]
Independent Events

\[ P(E_1 \cap E_2 \cap E_3) = P\{E_1\} \cdot P\{E_2\} \cdot P\{E_3\} \]
Applications in reliability

- Example 2.18
- Example 2.19
- Example 2.20
Conditional probability

• Updating of the sample space based on new information

• Consider two events $A$ and $B$. Suppose that the event $B$ has occurred. This information will change the probability of event $A$.

• $P(A|B)$ denotes the conditional probability of event $A$ given that $B$ has occurred.
Conditional probability

• If $A$ and $B$ are events in $\Omega$ and $P(B)>0$, then $P(A|B)$ is called the conditional probability of $A$ given $B$ if the following axiom is satisfied:
  \[ P(A|B) = \frac{P(A \cap B)}{P(B)} \]

• Example: tossing a fair dice.
  – $A = \{\text{the number on the dice is even}\}$
  – $B = \{\text{the number on the dice < 4}\}$
  – $P(A|B) = ?$
Independence

• If $P(A|B)=P(A)$ we call that event $A$ is independent of event $B$

• Note:
  – if two events $A$ and $B$ are independent, then
    $$P(A \cap B) = P(A)P(B)$$

• Show that $P(B|A)=P(B)$ also holds in this case.
  – In other words, $A$ and $B$ are mutually independent

• This does NOT mean that they are disjoint. If $A$ and $B$ are disjoint then $P(B|A)=0$
Independence

• Example: tossing a fair dice.
  – \( A = \{ \text{the number on the dice is even} \} \)
  – \( B = \{ \text{the number on the dice > 2} \} \)
  – \( P(A|B) = ? \)
  – \( P(B|A) = ? \)
  – \( P(A) = ? \)
  – \( P(B) = ? \)

• Example 2.31
Bayes’ Rule

• Using conditional probability formula we may write:
  – \( P(A|B) = \frac{P(A \cap B)}{P(B)} \)
  – \( P(B|A) = \frac{P(A \cap B)}{P(A)} \)
  – \( \rightarrow P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) \rightarrow P(B|A) = \frac{P(A|B)P(B)}{P(A)} \)

• This is known as the Bayes’ rule
• It forms the basis of Bayesian statistics
• What additional probabilities do we need to know to solve Example 2.32?
Law of Total Probability

• Let $B_1, B_2, B_3, \ldots, B_k$ be a partition of the sample space. $B_i$s are mutually disjoint. Let $A$ be any event.

• Note that $B_i$s also partition $A$

• Then for each $i = 1, 2, \ldots, k$

$$P(B_i \mid A) = \frac{P(A \mid B_i) P(B_i)}{P(A)} = \frac{P(A \mid B_i) P(B_i)}{\sum_{j=1}^{k} P(A \mid B_j) P(B_j)}$$

When $P(A)$ is not directly known, but known conditionally, we make use of this law.
Bayes’ Rule for two events

\[ P(B \mid A) = \frac{P(A \mid B)P(B)}{P(A)} = \frac{P(A \mid B)P(B)}{P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})} \]

• Now, solve Exercise 2.32, given \( P(B) \)
Another example

• A novel disease diagnostic kit is 95% effective in detecting a certain disease when it is present. The test also has a 1% false positive rate. If 0.5% of the population has the disease, what is the probability a person with a positive test result actually has the disease?
Solution

• $A = \{\text{a person’s test result is positive}\}$
• $B = \{\text{a person has the disease}\}$
• $P(B) = 0.005$, $P(A|B) = 0.95$, $P(A|B^c) = 0.01$

\[
P(B|A) = \frac{P(A|B)P(B)}{P(A|B)P(B) + P(A|B^c)P(B^c)}
\]

\[
= \frac{0.95 \times 0.005}{0.95 \times 0.005 + 0.01 \times (1 - 0.005)} = \frac{475}{1470} \approx 0.323
\]
Random Variables

• A random variable (r.v.) associates a unique numerical value with each outcome in the sample space. It is a real-valued function from a sample space $\Omega$ into real numbers.

• Similar to events it is denoted by an uppercase letter (e.g., $X$ or $Y$) and a particular value taken by a r.v. is denoted by the corresponding lowercase letter (e.g., $x$ or $y$).
Examples

• Toss three coins. $X = \text{number of heads}$
• Pick a student from the Computer Engineering Department.
  $X = \text{age of the student}$
• Observe lifetime of a light bulb.
  $X = \text{lifetime in minutes}$
• $X$ may be discrete or continuous